

An Entropic Formulation of Tunneling Time

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(Dated: December 29, 2015)

Quantum tunneling governs numerous phenomena in biology, chemistry, physics and technology. Tunneling time, formulated in various different forms due to the absence of a time operator in quantum theory, has been measured recently in experiments based on the attoclock in ultrafast laser ionization of Helium atoms [A. Landsman *et al.*, *Optica* **1**, 343 (2014)]. The experiment performs a refined measurement with which no tunneling time formula in the literature exhibits adequate congruence. Here we show that, entropic considerations lead to a real tunneling time which shows remarkable agreement with the experimental data and stays always subluminal. Indeed, with phase space volume setting the number of microstates for a single evanescent particle in a state of definite momentum consistent with energy conservation, one is naturally led to a statistical description for quantum tunneling in which thermal energy sets the tunneling time. This entropic tunneling time is rather general and might also be extended to photon and phonon tunneling phenomena.

PACS numbers: 03.65.Xp, 03.65.Ca, 05.30.-d

I. INTRODUCTION

Tunneling, passage of subatomic particles through the regions of space forbidden to classical motion, is a pure quantum phenomenon. Its physical relevance was first established by Gamow in his analysis of the α -decay [1, 2]. The tunnel diode [3] of Esaki was its first technological application. Undoubtedly, scanning tunneling microscope (STM) [4] of Binnig and Rohrer started a new pace in scientific and technological advancements. Tunneling is a ubiquitous mechanism that underlies numerous physical [5], chemical [6], biological [7] and technological events [8].

Tunneling time, the time elapsed during the tunneling process, is crucial for determining reaction speeds of tunneling-enabled rare processes ranging from high-speed electronic devices to nuclear fusion. In fact, with the advent of strong laser ionization experiments [9, 10], it is becoming possible to measure the tunneling time [11, 12] where certain metrological problems [13, 14] with the detection of the tunneling particle are shown to be surmountable [15, 16]. Strong laser fields enable electrons to tunnel out of atoms, where the potential barrier formed forms a testbed for models of tunneling time [17].

Tunneling is a pure quantum phenomenon. Tunneling time, however, is not a quantum concept [19]. Time, in general, is a deterministic dimension in both classical and quantal realms. The reason is that time cannot be an observable as otherwise it would be representable by an operator whose eigenstates would have time flow completely halted. Time is therefore a deterministic parameter underlying the indeterminism built in quantum theory. Tunneling time depends on what kinetic theory is set forth for tunneling process, and therefore, the literature consists of various time definitions [20]. They include traversal time through modulated barriers [21–23], spin precession time [24, 25], flux-flux correlation

duration [26], phase stationarity time [27, 28], and Feynman path integral (FPI) averaging of the classical time [29]. Some of them are complex, some are difficult to associate with tunneling and some suffer from superluminality. Interestingly, contrary to their *raison d'être*, all these tunneling times utilize a sort of time operator since they involve derivatives with respect to energy or potential. This is not the case for FPI averaging yet the resulting time is still controversial because classical trajectories live in imaginary time and their probability amplitudes interfere [30, 31]. At present, the problem with these and other tunneling time definitions is that they are simply incapable of explaining the experimental data as was comparatively analyzed and experimented in [32]. In view of the growing scientific and technological needs, however, it is necessary to have a working model that can reliably estimate the tunneling time for a given potential barrier.

The present work reports on a novel formulation of the tunneling time. The constructed time formula reproduces mean of the experimental data and precludes possibility of superluminal tunneling. The formulation, based on a statistical description of the evanescent particle in the classically forbidden region, gives a tunneling time which is remarkably accurate compared to other widely-used time definitions. The essence of the formulation is that, in the classically forbidden region time flows in imaginary direction, and correspondence between imaginary time in quantum mechanics and temperature in statistical mechanics enables a statistical formulation for tunneling. The resulting thermal energy, through the uncertainty principle, sets a time interval, which works through as a realistic model of tunneling time with the verification provided by the most recent experimental data [32], with the preclusion of superluminality and instantaneity, and with the derivation from a statistical viewpoint. The constructed tunneling time formula ap-

plies to all tunneling-enabled phenomena described by a time-independent potential barrier.

In Sec. 2 below given are derivation of the entropy characterizing the tunneling particle and definition of the tunneling time. Sec. 3 is devoted to a detailed test of the entropic tunneling time with the available experimental data. In Sec. 4, the entropic tunneling time is contrasted with other tunneling time definitions and also it is shown to permit no superluminal transition. Sec. 5 concludes the work and gives future prospects on applications to different tunneling-enabled phenomena and extensions to photon and phonon tunnelings.

II. ENTROPIC TUNNELING TIME

In classical dynamics, time elapsed while a particle moves from one point to another cannot be determined without knowing its momentum at each point in between. This is because momentum is the generator of translations and, with strict energy conservation, it becomes $\sqrt{2m(E - V(x))}$ for a particle moving along x axis with mass m , potential energy $V(x)$ and total energy E . The region bounded by x_L and $x_R > x_L$ at which $V(x_L) = E = V(x_R)$ and between which $E < V(x)$ is forbidden to classical motion. The particle returns to its region of incidence from the classical turning points x_L and x_R . This forbidden region accommodates only those particles having negative kinetic energy ($E - V(x) < 0$) and hence imaginary momentum ($\sqrt{2m(E - V(x))} \in \Im$). Time in this region, thus, flows purely in imaginary direction as $t \rightarrow -i\tau_c$ (see [33] for time arrow) such that

$$\tau_c = \int_{x_L}^{x_R} \frac{m dx}{\wp(x)} \quad (1)$$

is the classical tunneling time, and

$$\wp(x) = \sqrt{2m(V(x) - E)} \quad (2)$$

is the momentum of the particle. In consequence, traversing the classically forbidden region appears to cost no real time. The scattering process is instantaneous and hence violates causality. Physically, however, tunneling time must be real and finite [34] as was discussed in [35] previously.

Tunneling is a pure quantum phenomenon. Tunneling time, however, is not a quantum concept [19]. The reason is that time cannot be an observable representable by some operator since then time would stop passing by in its eigenstates. It is a deterministic dimension. Physically correct description of tunneling time might therefore involve an amalgamate of the classical description above and quantum behavior. First, as a fact inferred from momentum dependence of the elapsed time in classical dynamics, tunneling time must be addressed not in the energy eigenstates $\psi_e(x)$ but in momentum eigenstates $\psi_m(x)$. The reason is that Schroedinger equation refers to energy not displacement of the particle. It is

thus momentum operator and its eigenstate $\psi_m(x)$ that pertain to tunneling time. Second, as a means of ensuring penetration of the particle into the classically forbidden region, the classical momentum $\wp(x)$ must function as the eigenmomentum associated with $\psi_m(x)$ to give

$$\frac{d}{dx}\psi_m(x) = -\frac{\wp(x)}{\hbar}\psi_m(x) \quad (3)$$

in agreement with the conservation of energy. This equation can always be integrated to find

$$\psi_m(x) = \psi_m(x_L) \exp\left\{-\frac{1}{\hbar} \int_{x_L}^x \wp(\tilde{x}) d\tilde{x}\right\} \quad (4)$$

which is an evanescent wave that decays exponentially as the particle penetrates farther and farther from x_L . This evanescent behavior is the key aspect of the tunneling phenomenon. It encodes all the essential ingredients needed to describe the tunneling dynamics. Physically, thus, it suffices to work with the conditional amplitude

$$\frac{\psi_m^\dagger(x_R)\psi_m(x_R)}{\psi_m^\dagger(x_L)\psi_m(x_L)} \quad (5)$$

whose absolute square gives the probability measure

$$p_m = e^{-2\Phi} \quad (6)$$

which involves only the abbreviated action

$$\Phi = \frac{1}{\hbar} \int_{x_L}^{x_R} \wp(x) dx \quad (7)$$

measured in units of \hbar . The conditional probability p_m vanishes for infinitely wide and infinitely high potential barriers ($p_m \rightarrow 0$ as $\Phi \rightarrow \infty$), and equals unity if the barrier is absent ($p_m \rightarrow 1$ as $\Phi \rightarrow 0$). It measures the strength of correlation between the probabilities of finding the particle at x_L and x_R . Nevertheless, pertaining to a definite momentum state, it cannot tell whether tunneling has really been completed or not. The question of whether the particle has tunneled or reflected is actually answered by the tunneling transmission probability p_t not p_m . It is obtained by solving the Schroedinger equation

$$\frac{d^2}{dx^2}\psi_e(x) = \left(\frac{\wp(x)}{\hbar}\right)^2 \psi_e(x) \quad (8)$$

wherein the energy eigenfunction $\psi_e(x)$, unlike the momentum eigenfunction $\psi_m(x)$, involves both right-evanescent and left-evanescent waves, which give rise to transmission and reflection probabilities. The transmission probability takes the form [36]

$$p_t = \frac{1}{\cosh^2 \Phi} \quad (9)$$

after solving (8) in the WKB approximation and matching the solutions at x_L and x_R . The WKB approximation scheme holds for smooth potentials, which are not atypical for tunneling-enabled phenomena [5–8].

The remaining message of classical dynamics is that, in the classically forbidden region, time is pure imaginary by necessity not by technicality of Wick rotation to Euclidean space as a mathematical tool. The imaginary time, when interpreted as inverse temperature, is known to transform propagators in quantum mechanics into partition functions in statistical mechanics [37]. This ensures that the tunneling time must necessarily be addressed in a statistical framework despite the peculiarity that what is referred to here is a single particle not a collection of particles as in statistical thermodynamics. Statistical approach necessitates specifying the number of microstates underlying the tunneling process which involves a single evanescent particle in a state of definite momentum consistent with energy conservation. Obviously, the number of microstates must be linear in Φ . It is so because the abbreviated action

$$\int_{x_L}^{x_R} \varphi(x) dx \quad (10)$$

of the particle is nothing but the volume of its phase space and, as follows from its definition in (7), Φ counts the total number of quantum actions \hbar contained in that volume. This number, not necessarily an integer because of the quantum uncertainty, is revealed also by the Bohr-Sommerfeld quantization rule for the abbreviated action of periodic motion since, in imaginary time, the potential barrier is effectively inverted. The total number of microstates is to be $1 + 2\Phi$ because entropy must vanish when the barrier is absent ($\Phi \rightarrow 0$). In consequence, the particle is ascribed the entropy [38]

$$S(p_m) = k_B p_m \log(1 - \log p_m) \quad (11)$$

after relating to p_m via (6). Obviously, $S(0) = 0$, $S(1) = 0$ and $S(p_m) \geq 0$, as they should. Needless to say, this loglog structure is an artefact of the exponential relationship between Φ and p_m . Indeed, use of the uniform probability $p_u = 1/(1 + 2\Phi)$ instead of (6) would result in the more familiar Boltzmann entropy $-k_B p_u \log p_u$ instead of (11).

The entropy formula (11) is based mainly on the fact that the number of microstates pertaining to a single tunneling particle is $1 + 2\Phi$, which means that $S(p_m) \cong k_B \Phi$ in quantum domain where $|\Phi| \lesssim 1$. This relationship between entropy and (Euclidean) action is a well-known result in quantum gravity, in particular, black holes [39]. In consequence, the entropy formula (11) can be regarded as a quantum-theoretic property holding thanks to imaginary-time, purely quantal, evanescent tunneling dynamics.

In this statistical formulation, internal energy equals the energy E of the particle. The thermal energy, on the other hand, equals the entropy rate of change of the internal energy. This is so because tunneling involves a single particle with quantum probabilistic qualities, and the energy rate of change of the tunneling entropy in (11)

gives

$$\frac{1}{k_B T} = -\frac{2\tau_c}{\hbar} p_m \left[\frac{1}{1 - \log p_m} + \log \frac{1}{1 - \log p_m} \right] \quad (12)$$

where τ_c is the classical tunneling time in (1). It is not surprising that the temperature T is proportional to the reciprocal of τ_c [37]. This temperature, defined for a single particle owing to its quantum indeterminacy, involves both the Boltzmann constant k_B and Planck constant \hbar . In a true thermodynamical system, whose quantum corrections always involve \hbar , there remains no \hbar sensitivity in classical limit. The special thing about tunneling is that it does not have any classical limit and hence its characterization involves both k_B and \hbar .

The energy E of the particle is strictly constant throughout its motion inside and outside the classically forbidden region. It is known with certainty. In contrast to E , however, the thermal energy $k_B T$ varies with potential landscape as in (12) and, physically, it sets a finite time interval Δt in the philosophy of the energy-time uncertainty product. This time interval, as insured by construction of the probability p_m in (6), must be nothing but the time elapsed while the particle gets from x_L to x_R . One here notes that these turning points vary with E and, in general, lower the E larger the $x_R - x_L$ and longer the tunneling duration. Thus, Δt has every reason to qualify as the tunneling time provided that the tunneling transition is completed within the time interval Δt . Thus, one defines tunneling time as

$$\Delta t = \frac{\hbar}{2\Delta E_{ther}} \quad (13)$$

which is no different than the energy-time uncertainty relation. In here, ΔE_{ther} is the thermal energy needed for completing the tunneling, and it reads as

$$\Delta E_{ther} = p_t (2\pi k_B T) \quad (14)$$

where $2\pi k_B T$ is the splitting between Matsubara levels [40] and p_t is the tunneling transmission coefficient defined in (9). At last, the tunneling time in (13) takes the form

$$\Delta t = -\frac{\tau_c}{2\pi} \cosh^2 \Phi e^{-2\Phi} \left(\frac{1}{1 + 2\Phi} + \log \frac{1}{1 + 2\Phi} \right) \quad (15)$$

for smooth potentials admitting the WKB approximation. The tunneling time formula (13), whose WKB form is given in (15), will be hereon called *entropic tunneling time* to distinguish it from other tunneling time definitions in the literature.

III. EXPERIMENTAL CONFIRMATION

Having done with modeling, it is now time to confront the entropic tunneling time with experiment. Electric fields of high-intensity lasers reshape Coulomb potential in atoms to form a potential barrier through which electrons can tunnel to continuum [17]. At the peak value \mathcal{E}

of the electric field, one of the electrons in He possesses the potential energy [18]

$$V(x) = -\frac{Z + a_1 e^{-a_2 x} + a_3 x e^{-a_4 x} + a_5 e^{-a_6 x}}{x} - \mathcal{E}x \quad (16)$$

wherein $Z = 1$, $a_1 = 1.231$, $a_2 = 0.662$, $a_3 = -1.325$, $a_4 = 1.236$, $a_5 = -0.231$, and $a_6 = 0.480$ in atomic units. This potential, the so-called SAE potential [18], pertains to a single active electron that ionizes off the He atom by tunneling.

The electron energy E remains unchanged up to second order in \mathcal{E} by symmetry reasons. Advancements in ultrafast science, where strong laser fields are used to ionize atoms by quantum tunneling, are capable of observing tunneling transition and measuring the tunneling time [9–12]. In spite of various factors affecting the experiments [13–16], improving on previous single-particle tunneling time measurements [11] by using attoclock in strong laser fields [12], in 2013 Landsman and her collaborators have performed a refined measurement of the tunneling time of electrons in He atom [32]. Their measurements have been shown to be stable [41] (see also the simulation study [42]) against non-adiabatic effects [13, 14]. This experiment thus forms a state-of-the-art testbed for all tunneling time models, including the entropic tunneling time (15).

In the experimental setup, laser intensity 3.478×10^{16} W/cm² \mathcal{E}^2 is varied from 0.730×10^{14} W/cm² to 7.50×10^{14} W/cm² by varying the peak electric field \mathcal{E} from 0.046 to 0.147 in atomic units (curiously, experiment reports results at $\mathcal{E} = 0.04$, too). The electron energy $E = -0.904$ a.u. is the first ionization energy of He. Momentum distribution of the liberated electrons (which must peak around zero momentum if detector is near x_R) are obtained by cold-target recoil-ion momentum spectrometer (COLTRIMS) and by velocity map imaging spectrometer (VMIS) (see the experiment section of [32] for details). The VMIS is used particularly at low laser intensities. Quantum tunneling is ensured to be the dominant ionization mechanism by keeping the Keldysh parameter [17] small ($\gamma < 2$). The experimental results are given in Fig. 3 of [32]. Depicted in Fig. 3 (a) are different tunneling times [21, 24, 26, 27, 29] contrasted with experiment's own results. Given in Fig. 3 (b) and (d) are tunneling times as functions of the peak electric field \mathcal{E} and barrier width (approximated as E/\mathcal{E} a.u. in the experiment). The experiment (as well as [41]) concludes that among all widely-used tunneling time definitions only the FPI time comes closest to its measurements (see Fig. 3 (b) and (d)).

For confronting the entropic tunneling time model with experiment, it suffices to replace the potential energy (16) in Φ and τ_c , and evaluate them with the turning points x_L and $x_R > x_L$, which are the roots of the vanishing kinetic energy condition

$$V(x) - E = 0 \quad (17)$$

for each value of the peak electric field \mathcal{E} . Increasing \mathcal{E} from 0.04 a.u. to 0.11 a.u., x_L increases from 1.252 a.u.

to 1.405 a.u., and x_R decreases from 21.195 a.u. to 6.794 a.u.. Correspondingly, classical tunneling time τ_c decreases from 749.772 as to 272.038 as. At these extremes, the entropic tunneling time Δt decreases from 101.641 as to 19.305 as. These numbers affirm feasibility of the attoclock experiments for measuring tunneling time.

The entropic tunneling time (15) is plotted in Fig. 1 as a function of the peak electric field \mathcal{E} and experiment's barrier width E/\mathcal{E} (which is not the true barrier width $x_R - x_L$). The entropic tunneling time Δt , shown by filled triangles in Fig. 1, is superimposed on Fig. 3 (b) and (d) of [32] (digitized by using [43]). The figure manifestly shows that the entropic tunneling time exhibits remarkable agreement with the experimental data. Indeed, the numerical values predicted by (15) closely follow the measured time values in [32]. On the theory side, the FPI time [29, 31], the theoretical prediction coming closest to the experimental data, exhibits good agreement with the experimental data in both panels. Yet, in the left panel, its predictions diverge from the data as the peak electric field increases. In the right panel, it matches with the COLTRIMS data at low barrier widths while it diverges at larger barrier widths. In contrast to these divergent behaviors in FPI time, the entropic tunneling time stays congruent to experimental data for a wide range of potential parameters. In conclusion, the entropic tunneling time in (15) outperforms all the widely-used tunneling time models (those shown in Fig. 3 (a) of [32] and the FPI time).

IV. COMPARISON WITH OTHER TIMES

Having proven the experimental confirmation of the entropic tunneling time, it is now time to contrast it with tunneling times in the literature.

The tunneling times in literature [20], as were comparatively studied in Fig. 3 (a) of [32], are far from explaining the experimental data. Among them, only the FPI time comes closest to the experimental data but even that falls short of the entropic tunneling time, as depicted in Fig. 1. The entropic tunneling time, with this experimental vantage point, becomes invulnerable if it proves physical in view of basic constraints and performs better than the tunneling times in literature.

There are already manifold time definitions in the literature [20]. They vary in their origins, formulations and predictions. Two of them, the Larmor and Buttiker-Landauer times, are special in that they are defined via not the potential $V(x)$ alone but with its purposive modifications. The Larmor time is based on the Larmor precession of spin when the classically forbidden region is covered by an external magnetic field [24, 25]. The Buttiker-Landauer time is defined via an oscillating barrier [20–23]. These two times have been argued [23, 44] to be not the means but deviations of the tunneling time distributions. They have obviously nothing in common

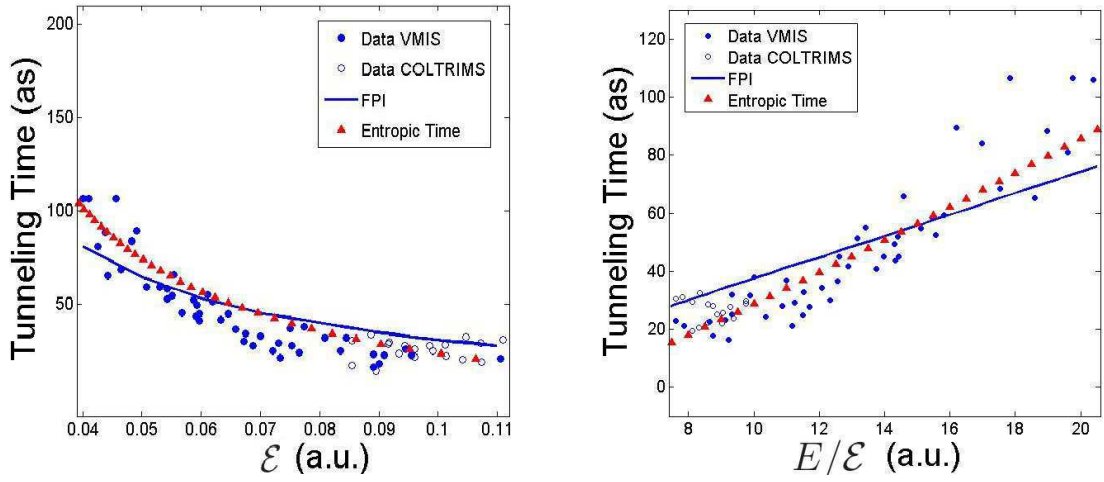


FIG. 1. Tunneling times as functions of peak electric field \mathcal{E} (left panel) and experimental barrier width E/\mathcal{E} (right panel) [43]. The entropic tunneling time Δt is depicted by filled triangles. It is seen to agree with the experimental data throughout. Both panels explicitly show how the entropic time adheres to the experimental data while the FPI time diverges away from data at the asymptotics.

with the entropic tunneling time, which is based on the potential barrier $V(x)$ alone.

The complex tunneling times are hard to make sense [45]. The path integral averages of the classical time [29] and of the Larmor time [46] give rise to complex times. They also arise via scattering-theoretic formulation [26]. Their real and imaginary parts are related to other tunneling times in specific ways [20, 23]. The entropic time is purely real and bears no relation to complex times.

There are two specific time definitions, phase time [20, 27, 28] and dwell time [20, 24], which, just like the entropic time, are based solely on the potential energy $V(x)$. They do not involve any external agents like the magnetic field in Larmor time. These three time definitions, having a common setup, can thus be directly contrasted to determine their physical relevance. In fact, without loss of generality, one can consider a simple rectangular potential barrier which forms an exactly soluble setup. Then, for a barrier of height V_0 and width L , one derives the tunneling transmission probability

$$p_t^\square = \frac{4\Phi^2\Phi_E^2}{-(\Phi^2 - \Phi_E^2)^2 + (\Phi^2 + \Phi_E^2)^2 \cosh^2 \Phi} \quad (18)$$

where

$$\Phi = \frac{\sqrt{2m(V_0 - E)}L}{\hbar} \quad (19)$$

is the abbreviated action in the classically forbidden region as follows from (7), and

$$\Phi = \frac{\sqrt{2mEL}}{\hbar} \quad (20)$$

is the one for free particle. The exact transmission probability (18) reduces to the WKB result in (9) only when $E = V_0/2$.

If the particle were obeying classical motion laws, it would traverse the barrier region within a time $-i\tau_c^\square$

where

$$\tau_c^\square = \frac{mL^2}{\hbar\Phi} \quad (21)$$

as follows from its definition in (1).

The entropic tunneling time, after using the transmission probability (18) in the thermal energy shift (14), takes the form

$$(\Delta t)^\square = -\frac{\tau_c^\square}{2\pi p_t^\square} e^{-2\Phi} \left[\frac{1}{1 + 2\Phi} + \log \frac{1}{1 + 2\Phi} \right] \quad (22)$$

which suffers from no approximate feature like the WKB transmission coefficient in (15).

The phase time, deriving from stationarity of the phase of the transmission amplitude [20, 27, 28], has the form

$$(\Delta t)_\varphi^\square = \frac{\tau_c^\square p_t^\square}{2\Phi^2\Phi_E^3} \left[\Phi\Phi_E^2 (\Phi^2 - \Phi_E^2) + (\Phi^2 + \Phi_E^2)^2 \sinh \Phi \cosh \Phi \right] \quad (23)$$

whose hyperbolic content arises from the phase of the transmission amplitude.

The dwell time, expressing how long the particle stays in the barrier region [20, 24], reads as

$$(\Delta t)_D^\square = \frac{\tau_c^\square p_t^\square}{2\Phi^2\Phi_E} \left[\Phi (\Phi^2 - \Phi_E^2) + (\Phi^2 + \Phi_E^2) \sinh \Phi \cosh \Phi \right] \quad (24)$$

involves lower powers of Φ and Φ_E compared to the phase time.

It is now time to contrast the three time definitions above. Though any value of Φ and Φ_E can be used in contrasting them, asymptotics prove particularly ef-

ficient. So indeed, it is straightforward to show that

$$(\Delta t)^\square \rightarrow \infty \quad (25)$$

$$(\Delta t)_\varphi^\square \rightarrow \frac{\hbar}{E} \sqrt{\frac{E}{V_0 - E}} \quad (26)$$

$$(\Delta t)_D^\square \rightarrow \frac{\hbar}{V_0} \sqrt{\frac{E}{V_0 - E}} \quad (27)$$

as $L \rightarrow \infty$. The entropic time diverges as expected of a potential barrier of infinite width. The phase and dwell times, however, give the unphysical result that it takes a finite time to traverse an infinitely wide potential barrier. These two times suffer from superluminality. More strikingly, those finite times vanish as $V_0 \rightarrow \infty$, meaning that the particle tunnels through an infinitely wide and infinitely high potential barrier instantaneously. This effect, the Hartman effect [47], renders the phase and dwell times completely unphysical. Needless to say, the entropic tunneling time is subluminal and possesses no unphysical features like the Hartman effect.

Rectangular potential barriers, apart from their direct solubility, prove useful in modeling tunneling systems whose potential barriers are nearly constant. The STM device is one such system [4]. And one finds that it takes $(\Delta t)^\square = 158.4$ as for an electron of energy $E = 1$ eV to traverse the tip-surface separation of $L = 1$ nm through the mean work function of $V_0 = 5$ eV. In the same setup, it would take much longer, $\tau_c = 843$ as, if tunneling were to proceed with classical dynamics alone. Here one recalls that τ_c is the absolute value of the imaginary time spent during the classical motion. The difference between the two tunneling times indicates how strong the quantum effects are. The qualitative consistency between Δt here and Δt computed-and-tested in the previous section ensures that strong quantum effects are properly taken into account by the entropic tunneling time formalism.

The other two times take the values $(\Delta t)_\varphi^\square = 329.12$ as and $(\Delta t)_D^\square = 65.82$ as in the same approximate STM setup. Numerically, one roughly estimates that $(\Delta t)_\varphi^\square \sim 2(\Delta t)^\square$ and $(\Delta t)_D^\square \sim (\Delta t)^\square/2$. All three times take thus femtosecond values. This rough numerical coincidence shows how crucial it is to examine a given tunneling time model at the asymptotics. Indeed, while all three times take femtosecond values at nano scales, only one of them, the entropic tunneling time, gives physically sensible results for opaque barriers. In fact, time measurements on semiconductor heterostructures, if accomplished with sufficient precision, can provide an independent examination for the entropic tunneling time in an exactly soluble setup.

V. TIME OUTSIDE THE BARRIER

The regions where $E > V(x)$ allow classical motion. They are situated in left of x_L and right of x_R to sandwich the classically forbidden region studied in Sec. 2.

The particle moves in these regions with positive kinetic energy ($E - V(x) > 0$) and hence real momentum ($\sqrt{2m(E - V(x))} \in \Re$) so that it needs time

$$t_c = \int_a^b \frac{m dx}{\sqrt{2m(E - V(x))}} \quad (28)$$

to arrive at b starting from $a < b$. This arrival time is purely real as it ought to be.

The momentum and energy eigenfunctions, as follow from (3) and (8) with $\varphi(x) \rightarrow -i\varphi(x)$ in view of (2), propagate instead of evanescing. This means that the probability amplitude that the particle has arrived at b given that it set out at a , defined similar to the probability amplitude (5), invariably leads to the conditional probability $p_m = 1$. Moreover, $p_m = 1$ also at the top of a rectangular barrier, that is, $E = V_0$. Thusly, the tunneling entropy (11) identically vanishes in classically allowed regions and the entropic tunneling time formalism of Sec. 2 reaches its validity limit. This is not surprising at all because time is purely real in these regions and there exists no reason for introducing a statistical formulation in which entropy production defines a time interval.

One thus concludes that, in classically allowed regions, time it takes for a particle to get from here to there must be analyzed not by statistical methods but by quantum-theoretic methods. Barring problems with time operator [19], one can devise, for example, certain compound operators of time dimension and require their expectation values to give the arrival time of the particle. As a plausible option, one can promote the classical time t_c in (28) to an arrival time operator and explore its physics implications by using Weyl quantization [48] or polymer quantization [49]. It is clear that the arrival operator is not the only option; one can of course invent different hermitian operators [50] to measure travel time of the particle.

Instead of operators, one can approach to the problem via path integral quantization. To this end, the approach of Sokolovski and Baskin in [29] proves particularly useful in that the classical time in (28) is averaged over all possible paths going from a to b . This procedure, though returns complex times, enables direct quantization and reproduces the Larmor times [45, 46].

Tunneling is a pure quantum effect with no classical substrate. This means that there exists no single path around which action of the particle stays stationary and thereupon Newton's equations follow. It is for this reason that construction of tunneling time necessitates an unorthodox formulation. In contradistinction with tunneling time, however, the arrival time outside the barrier has a well-defined classical support thanks to validity of Newton's equations in these regions and it is expected to take a value around the classical time t_c in (28). Thus, despite the absence of an authoritative quantum-theoretic framework, the arrival time can be ascribed at least an expected value around t_c – a property which never holds in the classically forbidden tunneling region.

VI. CONCLUSION AND FUTURE PROSPECTS

The entropic tunneling time, with its statistical conception, subluminal nature and experimental confirmation, works through as a realistic model of the tunneling time. It provides a quantum theoretic framework by which one can analyze all kinds of tunneling-enabled phenomena ranging from Alpha decay to STM, DNA mutation to enzymes, and flash memory to interstellar chemistry. This rather widespread role facilitates phenomenological tests and possible improvements of the entropic formalism through a variety of sources. Tunneling times of certain selected phenomena will be studied elsewhere [51].

The tunnel effect, a manifestation of the evanescent wave behavior, can occur in all wave phenomena. A generic wave equation

$$\frac{d^2}{dx^2}W(x) = -k^2(x)W(x) \quad (29)$$

portrays a propagating wave for $k(x) \in \Re$ and evanescent wave for $k(x) = i\kappa(x) \in \Im$. Pragmatically, the entropic tunneling time formalism can be extended to this wave behavior with the identification

$$\frac{\wp(x)}{\hbar} \rightarrow \kappa(x) \quad (30)$$

as revealed by contrasting (29) with (8). This is a formal equivalence, however. It is crucial to go deeper for it to make sense. First, it is necessary to determine the origin of the imaginary, inhomogeneous wavenumber $\kappa(x)$ in view of (2). Indeed, monochromatic wave must have a frequency below the natural cutoff frequency of the

medium for evanescent behavior to occur. Next, one has to construct the quanta corresponding to the wave so that evanescent characterizes tunneling. Last, analogy with the Schrodinger equation must be implemented by taking into account the symmetries of the wave equation.

There are numerous wave phenomena. The probability waves of quantum theory, $W(x) \equiv \psi_e(x)$, govern electron tunneling in semiconductors, Hydrogen tunneling in biochemical systems and Helium tunneling in nuclear systems. The electric waves, $W(x) \equiv \mathcal{E}(x)$, describe photon tunneling in materials with imaginary refractive index (band gaps, dielectric gaps, air gaps) [52]. The photonic STM [53], scanning of surfaces with a fiber optic tip, is a direct application of photon tunneling. The sound waves, on the other hand, encode phonon tunneling through acoustic band gaps [54]. Tunneling of the thermal vibrations of an STM tip to the sample is a direct realization of the phonon tunneling [55]. The optical and acoustic tunneling studies have been thoroughly reviewed in [56] experiment by experiment. The photon and phonon tunneling processes, interpreted so far only with phase and dwell times [56], need be analyzed and reinterpreted within the entropic tunneling time formalism, as is being planned to be done in upcoming work.

The entropic tunneling time is new. It is theoretically consistent and experimentally pertinent. It can qualify as a full-fledged tunneling time formalism only after experimental tests through different phenomena and different applications.

The authors are grateful to Onur Rauf Yilmaz for reading the manuscript and commenting on variations of entropy.

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